An analysis for the delta function with support on the light cone

1988 J. Phys. A: Math. Gen. 21 2667

(http://iopscience.iop.org/0305-4470/21/12/011)

J. Phys. A: Math. Gen. **21** (1988) 2667-2675. Printed in the UK

An analysis for the delta function with support on the light cone

R Estrada† and R P Kanwal‡

† Escuela de Mathemática, Universidad de Costa Rica, San José, Costa Rica
‡ Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

Received 1 September 1987, in final form 21 December 1987

Abstract. We study the delta function $\delta\langle\xi, \xi\rangle$, where ξ is a point in the Minkowski space and quantity $\langle\xi, \xi\rangle = t^2 - x_1^2 - x_2^2 - x_3^2 = t^2 - r^2$. This generalised function has its support on the light cone $t \pm r = 0$, which has an interesting singularity at its vertex $(0, \mathbf{0})$. We present the derivatives of the multilayers spread over this cone. Our analysis is facilitated by the interplay between the theory of distributions (generalised functions) and the theory of singular surfaces.

1. Introduction

The aim of the present work is to present an appropriate framework that permits us to study various generalised functions with support on the light cone $t \pm r = 0$ and to obtain the formulae for general partial derivatives of such generalised functions.

Formulae for the derivatives of functions discontinuous across a regular surface as well as the corresponding analysis of the associated multilayers have been obtained by several authors (Costen 1981, Estrada and Kanwal 1980, 1985a, b, 1987a, b, Vladimirov 1971). In particular, the general formulae for the arbitrary derivatives of multilayers are given by Estrada and Kanwal (1987a).

However, these formulae cannot be applied to the case of the light cone $t \pm r = 0$ because this spacetime surface is singular at the point $(0, \mathbf{0})$. Accordingly, the formulae valid for regular surfaces should be modified by the addition of suitable distributions concentrated at $(0, \mathbf{0})$, i.e. distributions of the form

$$\delta^{(k)}(t)\delta^{(m_1)}(x_1)\delta^{(m_2)}(x_2)\delta^{(m_3)}(x_3).$$

The action of certain partial differential operators, particularly the operator $\Box^2 = \nabla^2 - \partial^2\partial t^2$, on distributions such as $\delta(t-r)/r$ is well known (De Jager 1970, Jones 1982, Kanwal 1983). However, the extension of those results to general partial differential operators does not seem simple within the framework of this analysis. The situation is somewhat similar to that of the function $1/r^k$ for which the action of the operator ∇^2, and its powers, has been well understood for a long time, but for which the general partial derivatives were not obtained until recently (Blaive and Metzger 1984, Estrada and Kanwal 1985b).

In this paper we give a unified method for constructing generalised functions supported on the light cone $t \pm r = 0$. We do this as follows. Instead of considering functions $g(t, x)$ defined in the cone we consider functions of the form $f(t, y) = g(t, ty)$

defined in the cylinder $t \geq 0$, $|y| = 1$. Accordingly, the multilayers supported on the light cone take the form $f(t, y) d_n^P \delta(t-r)$, where f is a generalised function in the cylinder and d_n^P stands for the multilayer distribution. This procedure enables us to extend the formulae valid for regular surfaces and in particular to obtain the derivatives of such multilayers. An important feature of this analysis is that the computations are simple to carry out and easy to generalise by using the properties of the delta function instead of having to rely on just the basic definitions (Parrott 1986).

2. Basic definitions

Let the $\boldsymbol{x}$ space, $\boldsymbol{x} = (x_1, x_2, x_3)$, be denoted by $\mathbb{R}^3$ and let t denote the time. If $\boldsymbol{\xi} = (t, \boldsymbol{x})$, then the quantity

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = t^2 - |x|^2 = t^2 - r^2 = (t-r)(t+r) \tag{2.1}$$

is the square of the length in the Minkowski space from the origin $(0, \mathbf{0})$ to the point $\boldsymbol{\xi}$. Accordingly, the Dirac delta function $\delta(\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle)$ has the support on the light cone $t \pm r = 0$.

Recall the formula

$$\delta(f(x)g(x)) = \frac{1}{|f'(a)g(a)|}\,\delta(x-a) + \frac{1}{|f(b)g'(b)|}\,\delta(x-b) \tag{2.2}$$

where $f(x)$ is a function with simple zero at $x = a$ such that $f(a) = 0$, $f'(a) \neq 0$, and $g(x)$ is a function with simple zero at $x = b \neq a$ such that $g(b) = 0$, $g'(b) \neq 0$. This formula yields

$$\delta(\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle) = \delta((t-r)(t+r)) = \frac{\delta(t-r)}{2r} + \frac{\delta(t+r)}{2r} \tag{2.3}$$

if either t is fixed (and $t \neq 0$) or r is fixed (and $r \neq 0$).

We have recently presented an interplay between the theory of distributions and the theory of singular surfaces (Estrada and Kanwal 1980, 1985a, b, 1987a, b, Kanwal 1983). However, the formulae that we have presented in these references cannot be directly applied to the present situation because the cone $t \pm r = 0$ (the support of the delta function) has a singularity at the point $(0, \mathbf{0})$. But by crafting some additional analysis we can analyse the delta function $\delta(\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle)$ and its various derivatives. For this purpose we need the concept of multilayers on a surface of discontinuity $\Sigma(t, \boldsymbol{x})$ which in the present case is the light cone $t - r = 0$.

The basic distribution concentrated on a moving surface $\Sigma(t, \boldsymbol{x})$ is the delta function $\delta(\Sigma(t, \boldsymbol{x}))$ whose action on the test function $\phi(t, \boldsymbol{x})$ (the space of C^∞ test function with compact support) (Kanwal 1983) is

$$\langle \delta(\Sigma), \phi \rangle = \int_{-\infty}^{\infty} \int_{\Sigma(t,x)} \phi(t, \boldsymbol{x})\, \mathrm{d}S\, \mathrm{d}t \tag{2.4}$$

where $\mathrm{d}S$ is the surface element on Σ. The second surface distribution that we need in our discussion is the 'normal derivative' $\delta'(\Sigma)$ defined as

$$\delta'(\Sigma) = n_i \frac{\partial}{\partial x_i}(\delta(\Sigma)) \tag{2.5}$$

where n_i are the components of the unit normal vector $\boldsymbol{n}$ to σ, and we have used the summation convention.

Another closely related surface distribution is the normal derivative *operator* $d_n\delta(\Sigma)$ given by

$$\langle d_n\delta(\Sigma), \phi\rangle = -\int_{-\infty}^{\infty}\int_{\Sigma(t,x)} \frac{d\phi}{dn}\, dS(x)\, dt. \tag{2.6}$$

These three surface distributions are related as follows:

$$d_n\delta(\Sigma) = \delta'(\Sigma) - 2\Omega\delta(\Sigma) \tag{2.7}$$

where Ω is the mean curvature of Σ. The distributions $\delta(\Sigma)$ and $d_n\delta(\Sigma)$ display the single layer and double layer of charge of unit density.

The delta derivative $\delta/\delta t$ is defined as

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + G\frac{df}{dn} \tag{2.8}$$

and G is the normal speed of the surface $\Sigma(t, x)$. It is the derivative as apparent to a non-relativistic observer moving with the front Σ. Similarly

$$\frac{\delta f}{\delta x_i} = \frac{\partial f}{\partial x_i} - n_i\frac{df}{dn} \tag{2.9}$$

is the surface derivative with respect to the cartesian coordinates of the surrounding space (Kanwal 1983).

The use of the δ derivatives as given in (2.8) and (2.9) enable us to derive the first-order distributional derivatives of the simple layer $f\delta(\Sigma)$. Indeed,

$$\frac{\bar{\partial}}{\partial t}(f\delta(\Sigma)) = \frac{\bar{\delta} f}{\delta t}\delta(\Sigma) - Gf\delta'(\Sigma) = \left(\frac{\bar{\delta} f}{\delta t} - 2\Omega f\right)\delta(\Sigma) - Gfd_n\delta(\Sigma) \tag{2.10}$$

and

$$\frac{\bar{\partial}}{\partial x_i}(f\delta(\Sigma)) = \frac{\bar{\delta} f}{\delta x_i}\delta(\Sigma) + n_i f\delta'(\Sigma) = \left(\frac{\bar{\delta} f}{\delta x_i} + 2\Omega n_i f\right)\delta(\Sigma) + n_i f d_n\delta(\Sigma) \tag{2.11}$$

where the bar denotes the distributional derivative.

The general multilayer $fd_n^P\delta(\Sigma)$ is defined as

$$\langle fd_n^P\delta(\Sigma), \phi(t, x)\rangle = (-1)^P \int_{-\infty}^{\infty}\int_{\Sigma} f(t, x)\frac{d^P\phi}{dn^P}\, dS\, dt = (-1)^P\langle f\delta(\Sigma), d^P\phi/dn^P\rangle. \tag{2.12}$$

Note that d_n^P is not defined as the Pth power of the operator d_n.

In the present problem the surface $\Sigma(t, x)$ is the light cone $t - r = 0$ so that the basic surface distribution is $\delta(t - r)$:

$$\langle\delta(t-r), \phi(t, x)\rangle = \int_0^{\infty}\int_{t=r}\phi(t, x)\, dS(x)\, dt = \int_0^{\infty} t^2\int_{S_1}\phi(t, ty)\, dS(y)\, dt \tag{2.13}$$

where we have put $x = ty$, and S_1 is the sphere of radius unity in the x space $\mathbb{R}^3$. Similarly, the distribution $d_n^P\delta(t-r)$, $P \geq 1$, is

$$\langle d_n^P\delta(t-r), \phi(t, x)\rangle = (-1)^P\langle\delta(t-r), d^P\phi/dn^P\rangle. \tag{2.14}$$

Note that the normal derivatives are discontinuous at $(0, \mathbf{0})$. Fortunately, this discontinuity causes no problem since $d^P\phi/dn^P$ is integrable near $t \pm r = 0$.

3. Mathematical analysis

As in relation (2.13) it is convenient to use the coordinates (t, y) with $|y|=1$, $t \geq 0$ to describe the points of the forward cone $t=r$ by putting $x=ty$. This defines a map from the half-cylinder $S_1 \times [0, \infty)$ onto the cone. If ϕ is a member of the class $\mathscr{D}(\mathbb{R}\times\mathbb{R}^3)$ of test functions (Kanwal 1983), then its restriction to the cone gives rise to an element $\tilde{\phi}(t, y)=\phi(t, ty)$ of $\mathscr{D}(S_1\times[0,\infty))$. Actually, the same is true of each normal derivative $\mathrm{d}^P\phi/\mathrm{d}n^P$, because even though the function is discontinuous at $(0, \mathbf{0})$ the associated function in the cylinder is smooth. These considerations suggest that the appropriate multilayers $f d_n^P \delta(t-r)$ are those where $f \in \mathscr{D}'(S_1\times[0,\infty])$, and where $\mathscr{D}'$ is the space dual to $\mathscr{D}$. They are defined as

$$\langle f(t,y) d_n^P \delta(t-r), \phi(t,x)\rangle = (-1)^P \langle t^2 f(t,y), \mathrm{d}^P\phi(t,ty)/\mathrm{d}n^P\rangle \tag{3.1}$$

where the last operation takes place in $\mathscr{D}'(S_1\times[0,\infty])\times\mathscr{D}(S_1\times[0,\infty])$.

Let us now observe that the relation (3.1) immediately gives

$$\delta(t) d_n^P \delta(t-r) = \delta'(t) d_n^P \delta(t-r) = 0. \tag{3.2}$$

Next, we attempt to find an expression for $\delta^{(k)}(t) d_n^P \delta(t-r)$ for $k \geq 2$. For this purpose, we start with $\delta^{(k)}(t)\delta(t-r)$ by introducing the function $\Phi(t)$:

$$\Phi(t) = \int_{S_1} \phi(t, ty)\, \mathrm{d}S(y) \tag{3.3}$$

so that

$$\langle \delta^{(k)}(t)\delta(t-r), \phi(t,x)\rangle = \langle t^2\delta^{(k)}(t), \Phi(t)\rangle = \frac{(-1)^k k!}{(k-2)!}\Phi^{(k-2)}(\mathbf{0}). \tag{3.4}$$

But using the value (Estrada and Kanwal 1985b)

$$c_m = c_{m,3} = \int_{S_1} y_i^{2m}\, \mathrm{d}S(y) = \frac{4\pi}{2m+1} \tag{3.5}$$

we obtain

$$\Phi^{(q)}(0) = \sum_{j=0}^{[q/2]} \binom{q}{2j} c_j \frac{\partial^{q-2j}}{\partial t^{q-2j}} \nabla^{2j}\phi(0, \mathbf{0}) \tag{3.6}$$

where $[q/2]$ is the greatest integer less than or equal to $q/2$. Thus,

$$\delta''(t)\delta(t-r) = 8\pi\delta(t)\delta(x) \tag{3.7}$$

and more generally

$$\delta^{(k)}(t)\delta(t-r) = \frac{4\pi k!}{(k-2)!}\sum_{j=0}^{[(k-2)/2]}\binom{k-2}{2j}\frac{1}{(2j+1)}\delta^{(k-2-2j)}(t)\nabla^{2j}\delta(x). \tag{3.8}$$

A similar analysis yields

$$\delta^{k}(t) d_n^P\delta(t-r) = \frac{4\pi k!}{(k-2)!}\sum_{j=[(P+1)/2]}^{[(k-2+P)/2]}\binom{k-2}{2j-P}\left(\frac{1}{2j+1}\right)\delta^{(k+P-2-2j)}(t)\nabla^{2j}\delta(x). \tag{3.9}$$

In the next stage we need some geometric quantities associated with the cone $t-r=0$ and use the notation as given by Kanwal (1983). The symmetric surface tensor μ_{ij}:

$$\mu_{ij} = \delta n_i/\delta x_j$$

is the second fundamental form of the surface, where n_i is the unit vector and $\delta/\delta x_j$ stands for the surface differentiation. The symmetry of this tensor is preserved even with respect to time, since

$$\mu_{it} = \delta n_i/\delta t = \delta(-G)/\delta x_i = \mu_{ti}$$

where G is the normal speed of the surface. Furthermore, we denote by $\mu_{ij}^{(r)}$ the entries of the rth power of matrix $\boldsymbol{\mu}$, and set

$$\mu_{ij}^{(0)} = \delta_{ij} - n_i n_j$$

so that

$$\mu_{ij}^{(1)} = \mu_{ij} = \delta n_i/\delta x_j \qquad \mu_{ij}^{(2)} = \mu_{ik}\mu_{kj} \qquad \mu_{ij}^{(3)} = \mu_{ik}^{(2)}\mu_{kj} \ldots \mu_{ij}^{(r)} = \mu_{ik}^{(r-m)}\mu_{kj}^{(m)}.$$

Also

$$\mu_{tt} = -\delta G/\delta t.$$

The value of the mean curvature Ω is

$$\Omega = -\tfrac{1}{2}\mu_{ii}$$

where i is summed.

For the cone $t - r = 0$, the above-mentioned geometrical quantities take the following simple form:

$$\begin{aligned} &n_i = x_i/t \qquad \mu_{ij}^{(P)} = t^{-P}(\delta_{ij} - n_i n_j) \\ &\Omega = -1/t \qquad G = 1 \qquad \mu_{it} = \mu_{tt} = 0. \end{aligned} \tag{3.10}$$

Next, we use formulae (2.10) and (3.10) and get

$$\frac{\bar{\partial}}{\partial t}\left(f d_n^P \delta(t-r)\right) = -f d_n^{P+1}\delta(t-r) + \left(\frac{\delta f}{\delta t} + \frac{2f}{t}\right) d_n^P \delta(t-r). \tag{3.11}$$

Observe that the division $f(t, y)/t$ does not give a uniquely determined distribution since the general solution of the division problem contains an arbitrary multiple of $\delta(t)$. But this causes no problem because of relation (3.2). Similarly, from relations (2.11) and (3.10) we find that

$$\frac{\bar{\partial}}{\partial x_i}\left(f d_n^P \delta(t-r)\right) = f n_i d_n^{P+1}\delta(t-r) + \sum_{M=0}^{P} \frac{P!}{M!}\frac{1}{t^{P-M}}\left(\frac{\delta f}{\delta x_i} - \frac{2n_i f}{t}\right) d_n^M \delta(t-r). \tag{3.12}$$

Here also, we have to be careful because we again have the division problem of the form

$$\frac{1}{t^{P-M}}\left(\frac{\delta f}{\delta x_i} - \frac{2n_i f}{t}\right)$$

and such a problem gives rise to $P - M$ arbitrary constants. However, a moment's reflection will convince the reader that it really does not matter as to which solutions of the division problem are taken, as long as the solutions are consistent in the sense that if g_k is the solution of the division f/t^k that is chosen as the solution of the division problem then tg_k is the solution that should be taken for the division f/t^{k-1}.

Let us denote by $(1/t^k)d_n^P\delta(t-r)$ the distribution

$$Pf\left(\frac{H(t)}{t^k}\right) d_n^P \delta(t-r)$$

where *Pf* stands for the pseudofunction in the sense of the Hadamard finite part and where $H(t)$ is the Heaviside function. The time derivatives of $(1/t^k)d_n^P\delta(t-r)$ can be obtained from (3.11) as follows:

$$\frac{\bar{\partial}}{\partial t}\left[\left(\frac{1}{t^k}\right)d_n^P\delta(t-r)\right]=-\frac{1}{t^k}d_n^{P+1}\delta(t-r)+\left[\frac{\bar{\delta}}{\delta t}\left(\frac{1}{t^k}\right)+\frac{2}{t}\left(\frac{1}{t^k}\right)\right]d_n^P\delta(t-r)$$

$$=-\frac{1}{t^k}d_n^{P+1}\delta(t-r)+\left(\frac{2-k}{t^{k+1}}+\frac{(-1)^k\delta^{(k)}(t)}{h!}\right)d_n^P\delta(t-r)$$

or

$$\frac{\bar{\partial}}{\partial t}\left(\frac{1}{t^k}d_n^P\delta(t-r)\right)=-\frac{1}{t^k}d_n^{P+1}\delta(t-r)+\frac{2-k}{t^{k+1}}d_n^P\delta(t-r)$$
$$+\frac{(-1)^k4\pi}{(k-2)!}\sum_{j=[(P+1)/2]}^{[(k-2+P)/2]}\binom{k-2}{2j-P}\frac{\delta_{(t)}^{(k+P-2-2j)}\nabla^{2j}(x)}{2j+1} \qquad (3.13)$$

where we have used (3.9) and the relation (Jones 1982, Kanwal 1983, Lighthill 1957)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[Pf\left(\frac{H(t)}{t^k}\right)\right]=-kPf\left(\frac{H(t)}{t^{k+1}}\right)+\frac{(-1)^k\delta^{(k)}(t)}{k!}.$$

As a particular case we obtain

$$\frac{\bar{\partial}}{\partial t}\left(\frac{\delta(t-r)}{t}\right)=-\frac{1}{t}d_n\delta(t-r)+\frac{1}{t^2}\delta(t-r). \qquad (3.14)$$

Higher-order time derivatives can be obtained by repeated application of formula (3.13). As a special case we get

$$\frac{\bar{\partial}^2}{\partial t^2}\left(\frac{\delta(t-r)}{t}\right)=\frac{1}{t}d_n^2\delta(t-r)-\frac{2}{t^2}d_n\delta(t-r)+4\pi\delta(t)\delta(x) \qquad (3.15)$$

and more generally,

$$\left(\frac{\bar{\partial}}{\partial t}\right)^{2Q}\left(\frac{\delta(t-r)}{t}\right)=\frac{1}{t}d_n^{2Q}\delta(t-r)-\frac{2}{t^2}d_n^{2Q-1}\delta(t-r)+4\pi\sum_{j=0}^{Q-1}\frac{\delta^{(2Q-2j-2)}(t)\nabla^{2j}\delta(x)}{2j+1} \qquad (3.16a)$$

$$\left(\frac{\bar{\partial}}{\partial t}\right)^{2Q+1}\left(\frac{\delta(t-r)}{t}\right)$$
$$=-\frac{1}{t}d_n^{2Q+1}\delta(t-r)+\frac{1}{t^2}d_n^{2Q}\delta(t-r)+4\pi\sum_{j=0}^{Q-1}\frac{\delta^{(2Q-2j-1)}(t)\nabla^{2j}\delta(x)}{2j+1}. \qquad (3.16b)$$

A similar analysis yields

$$\left(\frac{\bar{\partial}}{\partial t}\right)^{P}\left(\frac{\delta(t-r)}{t^2}\right)=\frac{(-1)}{t^2}d_n^P\delta(t-r)+4\pi\sum_{j=0}^{[(P-1)/2]}\frac{\delta^{(P-2j-2)}(t)\nabla^{2j}\delta(x)}{2j+1}. \qquad (3.17)$$

Let us now consider the space derivatives. Using (3.12) we obtain

$$\frac{\bar{\partial}}{\partial x_i}\left(\frac{1}{t^k}d_n^P\delta(t-r)\right)=\frac{n_i}{t^k}d_n^{P+1}\delta(t-r)-2\sum_{M=0}^{P}\frac{P!}{M!}\frac{n_i}{t^{k+P-M+1}}d_n^M\delta(t-r). \qquad (3.18)$$

In particular,

$$\frac{\bar{\partial}}{\partial x_i}\left(\frac{\delta(t-r)}{t^k}\right)=\frac{n_i}{t^k}d_n\delta(t-r)-\frac{2n_i}{t^{k+1}}\delta(t-r). \qquad (3.19)$$

The mixed space and time derivatives can then be obtained by combining formulae (3.13) and (3.18). In order to do that it will be to our advantage to compute the value of the generalised function

$$n_i\delta^{(k)}(t)d_n^P\delta(t-r).$$

In the case $P=0$ and $\phi(t,x)$ is a test function of $\mathscr{D}(\mathbb{R}\times\mathbb{R}^3)$ we obtain

$$\langle n_i\delta^{(k)}(t)\delta(t-r),\phi(t,x)\rangle=\frac{(-1)^k k!}{(k-2)!}\Psi^{(k-2)}(0) \tag{3.20}$$

where

$$\Psi(t)=\int_{S_1}\phi(t,ty)y_i\,\mathrm{d}S(y). \tag{3.21}$$

But since

$$\Psi^{(q)}(0)=4\pi\sum_{j=0}^{[(q-1)/2]}\binom{q}{2j+1}\frac{\partial^{q-2j-1}}{\partial t^{q-2j-1}}D_i\nabla^{2j}\phi(0,\mathbf{0})\frac{1}{2j+3}$$

we obtain

$$n_i\delta^{(h)}(t)\delta(t-r)=\frac{4\pi k!}{(h-2)!}\sum_{j=0}^{[(k-3)/2]}\binom{k-2}{2j+1}\frac{\delta^{(k-3-2j)}(t)D_i\nabla^{2j}\delta(x)}{2j+3}. \tag{3.22}$$

We mention the special cases:

$$n_i\delta^{(k)}(t)\delta(t-r)=0 \qquad k\leqslant 2$$

$$n_i\delta'''(t)\delta(t-r)=8\pi\delta(t)D_i\delta(x).$$

A similar study gives

$$n_i\delta^{(k)}(t)d_n^P\delta(t-r)=\frac{4\pi k!}{(k-2)!}\sum_{j=[P/2]}^{[(k-3+P)/2]}\binom{k-2}{2j-P+1}\frac{\delta^{(k-3+P-2j)}(t)D_i\nabla^{2j}\delta(x)}{2j+3}. \tag{3.23}$$

Using (3.23) we readily obtain the mixed derivatives as

$$\begin{aligned}
&\frac{\bar{\partial}^2}{\partial t\,\partial x_i}\left(\frac{1}{t^k}d_n^P\delta(t-r)\right)\\
&\quad=-\frac{n_i}{t^k}d_n^{P+2}\delta(t-r)+\frac{(4-k)n_i}{t^{k+1}}d_n^{P+1}\delta(t-r)\\
&\qquad+2\sum_{M=0}^{P}\frac{(P+k-1)P!}{M!}\frac{n_i}{t^{k+2+P-M}}d_n^M\delta(t-r)\\
&\qquad+\frac{(-1)^k4\pi}{(k-2)!}\sum_{j=[(P+1)/2]}^{[(k-2+P)/2]}\binom{k-2}{2j-P}\frac{\delta^{(k+P-2-2j)}(t)D_i\nabla^{2j}\delta(x)}{2j+1}.
\end{aligned} \tag{3.24}$$

In particular,

$$\frac{\bar{\partial}^2}{\partial t\partial x_i}\left(\frac{\delta(t-r)}{t}\right)=-\frac{n_i}{t}d_n^2\delta(t-r)+\frac{3n_i}{t^2}d_n\delta(t-r). \tag{3.25}$$

The second-order space derivatives take the form

$$\frac{\bar{\partial}^2}{\partial x_i\,\partial x_j}\left(\frac{1}{t^k}\,d_n^P\delta(t-r)\right)$$

$$=\frac{n_in_j}{t^k}\,d_n^{P+2}\delta(t-r)+\frac{\delta_{ij}-5n_in_j}{t^{k+1}}\,d_n^{P+1}\delta(t-r)$$

$$+\sum_{M=0}^{P}\left(\frac{(2M-P-1)}{N!}(\delta_{ij}-3n_in_j)-\frac{2P!}{(M-1)!}\,n_in_j\right)\frac{1}{t^{k+P-M+2}}\,d_n^M\delta(t-r). \tag{3.26}$$

Putting $i=j$ and summing we get

$$\bar{\nabla}^2\left(\frac{1}{t^k}\,d_n^P\delta(t-r)\right)=\frac{1}{t^k}\,d_n^{P+2}\delta(t-r)-\frac{2}{t^{k+1}}\,d_n^{P+1}\delta(t-r)-\sum_{M=1}^{P}\frac{2P!d_n^M\delta(t-r)}{(M-1)!\,t^{k+P-M+2}}. \tag{3.27}$$

Special cases include

$$\frac{\bar{\partial}^2}{\partial x_i\,\partial x_j}\left(\frac{\delta(t-r)}{t}\right)=\frac{n_in_j}{t}\,d_n^2\delta(t-r)+\frac{(\delta_{ij}-5n_in_j)}{t^2}\,d_n\delta(t-r)-\frac{(\delta_{ij}-3n_in_j)}{t^3}\,\delta(t-r) \tag{3.28}$$

and

$$\bar{\nabla}^2\left(\frac{\delta(t-r)}{t}\right)=\frac{1}{t}\,d_n^2\delta(t-r)-\frac{2}{t^2}\,d_n\delta(t-r). \tag{3.29}$$

As a check on our formulae we subtract (3.29) from (3.15) and get

$$-\bar{\Box}^2\left(\frac{\delta(t-r)}{t}\right)=\left(\frac{\bar{\partial}^2}{\partial t^2}-\bar{\nabla}^2\right)\left(\frac{\delta(t-r)}{t}\right)=4\pi\delta(t)\delta(\boldsymbol{x}) \tag{3.30}$$

which is a well known formula.

The analysis for the second factor in (2.3) is similar. Thereby, we have completed a distributional analysis for delta functions concentrated on the light cone.

4. Summary

We have stressed the interplay between some concepts of generalised functions and differential geometry and have analysed the multilayers $f(t,y)d_n^P\delta(t-r)$ spread over the light cone. The vertex of the cone is a singular point on this surface. By a simple coordinate adjustment we have defined a map from the half-cylinder $S_1\times[0,\infty]$ onto the cone, where S_1 is the sphere of radius unity.

Various interesting derivatives such as $(\bar{\partial}/\partial t)^P(\delta(t-r)/t)$ and $n_i\delta^{(k)}(t)d_n^P\delta(t-r)$ have been presented.

References

Blaive V and Metzger J 1984 *J. Math. Phys.* **25** 1721-4
Costen R C 1981 *J. Math. Phys.* **22** 1377-85
De Jager E M 1970 *Mathematics Applied to Physics* ed E Roubine (Berlin: Springer) pp 52-110

Estrada R and Kanwal R P 1980 *J. Inst. Math. Appl.* **26** 39-63
—— 1985a *J. Math. Anal. Appl.* **89** 478-90
—— 1985b *Proc. R. Soc.* A **401** 281-97
—— 1987a *Circo. Mat. Palermo* in press
—— 1987b *J. Math. Anal. Appl.* **128** 389-404
Jones D S 1982 *Generalised Functions* (Cambridge: Cambridge University Press)
Kanwal R P 1983 *Generalized Functions, Theory and Techniques* (New York: Academic)
Lighthill M J 1957 *An Introduction to Fourier Analysis and Generalised Functions* (Cambridge: Cambridge University Press)
Parrott S 1986 *Relativistic Electrodynamics and Differential Geometry* (Berlin: Springer)
Vladimirov U S 1971 *Equations of Mathematical Physics* (New York: Dekker)